JOURNAL OF APPROXIMATION THEORY **55**, 318–325 (1988)

Some Characteristic and Non-characteristic Properties of Inner Product Spaces

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Communicated by Frank Deutsch

Received March 31, 1986; revised September 3, 1986

INTRODUCTION

Let E be a real or complex normed linear space with unit sphere $S=\{x\in E\colon \|x\|=1\}$ and let $\lambda>0$, $0<\varepsilon<2$. We say that E satisfies, respectively, the properties P_λ and R_ε if

$$P_\lambda\colon x,\ y\in S,\ \|x+\lambda y\|=\|x-\lambda y\| \Rightarrow \|x+\lambda y\|^2=1+\lambda^2,$$

$$R_\varepsilon\colon \delta(\varepsilon)=1-(1-\varepsilon^2/4)^{1/2},$$

where $\delta(\varepsilon)=\inf\{1-(\|x+y\|/2)\colon x,\ y\in S,\ \|x-y\|=\varepsilon\}$ denotes the modulus of convexity of E.

It is well known that inner product spaces satisfy the above properties for every λ and ε. On the other hand Borwein and Keener [4] and Nordlander [10] conjectured, respectively, that if E satisfies either P_λ or R_ε for some λ or ε then E is an inner product space.

Although the equivalence between the above properties is known (it is mentioned in [3]), we cannot give an exact reference for it and we commence this paper with a proof of this equivalence when $\lambda=\varepsilon(4-\varepsilon^2)^{-1/2}$. Then we prove that the mentioned conjectures are true for almost every λ and ε, but they are false (at least when E is real and two-dimensional) for λ and ε belonging to a countable and dense subset of $\mathbb{R}_+$ and $(0, 2)$, respectively. In particular, we prove that the conjecture is true for the case $\lambda=2$ specially considered in [4] in connection with some problems relative to Chebyshev centers.

With this and the paper of Amir and Mach [2] all the conjectures and open questions posed in [4] and [10] are solved, except for a new conjecture which can be stated in the following terms: either P_λ or R_ε, without restriction on the values of λ and ε, is a characteristic property of the inner product spaces of real dimension $\geqslant 3$.

0021-9045/88 $3.00

RESULTS

The following elementary lemma is the basis of many arguments which we shall use.

LEMMA 1. *Let S be the unit sphere of a norm in $\mathbb{R}^2$ and let $s(\alpha)$ be the point of S which is to a given point $s(0)$ at an angle $0 \leqslant \alpha < 2\pi$, measured with a given orientation of the plane. Then for every $\lambda > 0$ the real functions*

$$\alpha \in [0, \pi] \to \|s(0) + \lambda s(\alpha)\|, \qquad \alpha \in [0, \pi] \to \|s(0) - \lambda s(\alpha)\|$$

are continuous and, respectively, decreasing and increasing.

PROPOSITION 1. *Let S be the unit sphere of a real or complex normed linear space E, let $0 < \varepsilon < 2$, and let*

$$\delta(\varepsilon) = \inf\{1 - \|x + y\|/2 \colon x, y \in S, \|x - y\| = \varepsilon\}$$

be the modulus of convexity of E. Then for $\lambda = \varepsilon(4 - \varepsilon^2)^{-1/2}$ the following properties are equivalent:

$$P_\lambda\colon x, y \in S, \|x + \lambda y\| = \|x - \lambda y\| \Rightarrow \|x + \lambda y\|^2 = 1 + \lambda^2$$

$$P'_\lambda\colon x, y \in S, \|x + \lambda y\|^2 = 1 + \lambda^2 \Rightarrow \|x + \lambda y\| = \|x - \lambda y\|$$

$$Q_\varepsilon\colon x, y \in S, \|x - y\| = \varepsilon \Rightarrow \|x + y\|^2 = 4 - \varepsilon^2$$

$$R_\varepsilon\colon \delta(\varepsilon) = 1 - (1 - \varepsilon^2/4)^{1/2}$$

$$T_\varepsilon\colon \forall x, y \in S, [\|x - y\| - \varepsilon][\|x + y\|^2 - (4 - \varepsilon^2)] \leqslant 0.$$

Proof. We shall prove the six implications

$$P_\lambda \Rightarrow P'_\lambda \Rightarrow Q_\varepsilon \Rightarrow R_\varepsilon \Rightarrow Q_\varepsilon \Rightarrow T_\varepsilon \Rightarrow P_\lambda.$$

As we shall see the third and the fifth are obvious. The first is rather intricate, but we shall make use of it later.

($P_\lambda \Rightarrow P'_\lambda$). Assume by contradiction the existence of $x, y \in S$ such that

$$\|x + \lambda y\|^2 = 1 + \lambda^2 \neq \|x - \lambda y\|^2.$$

Then it follows from Lemma 1 and the hypothesis that in the real plane $\langle x, y \rangle$, endowed with the orientation $\omega = [x, y]$, there exist $u, v \in S$ such that $[u, y] = [x, v] = \omega$ and

$$\|u + \lambda y\|^2 = 1 + \lambda^2 = \|u - \lambda y\|^2, \qquad \|x + \lambda v\|^2 = 1 + \lambda^2 = \|x - \lambda v\|^2.$$

Let $a, b \in \mathbb{R}$ be such that $ax + b(u-x) = x + \lambda y$. Since the function $F(t) = \|ax + t(u-x)\|$ is convex and such that

$$F(0) = F(a) = |a|, \qquad F(b) = F(b+1) = (1+\lambda^2)^{1/2}$$

the ordering of the points 0, a, b, $b+1$ implies that F attains its minimum value, $(1+\lambda^2)^{1/2}$, in a real segment which contains such points.

If $a = -(1+\lambda^2)^{1/2}$ then, writing

$$-(1+\lambda^2)^{1/2}\, x + b(u-x) = x + \lambda y$$

in the equivalent form

$$-(1+\lambda^2)^{1/2}\, x + \frac{(1+\lambda^2)^{1/2}\, b}{1+(1+\lambda^2)^{1/2}}\,(u-x) = \frac{(1+\lambda^2)^{1/2}\,\lambda}{1+(1+\lambda^2)^{1/2}}\, y$$

and taking into account that

$$F\left(\frac{(1+\lambda^2)^{1/2}\, b}{1+(1+\lambda^2)^{1/2}}\right) = (1+\lambda^2)^{1/2},$$

we obtain the false equality

$$\frac{(1+\lambda^2)^{1/2}\,\lambda}{1+(1+\lambda^2)^{1/2}} = (1+\lambda^2)^{1/2}.$$

Therefore $a = (1+\lambda^2)^{1/2}$ and the four points

$$x,\ u,\ (1+\lambda^2)^{-1/2}\,(x+\lambda y),\ (1+\lambda^2)^{-1/2}\,(u+\lambda y)$$

are in a segment contained in S.

For analogous reasons also the four points

$$y,\ v,\ (1+\lambda^2)^{-1/2}\,(x+\lambda y),\ (1+\lambda^2)^{-1/2}\,(x+\lambda v)$$

are in a segment contained in S.

If $\|x-\lambda y\|^2 > 1+\lambda^2$ then $[x, u] = [v, y] = \omega$ and all the above seven points are in the straight line passing through x and y, from which follows the contradictory equality $\|x+\lambda y\| = 1+\lambda$.

In the other case, $\|x-\lambda y\|^2 < 1+\lambda^2$, we have that $\|u+\lambda v\|^2 < 1+\lambda^2$ and hence that for every $0 < t < 1$

$$\|tx + (1-t)u + \lambda[ty + (1-t)v]\|^2 < 1+\lambda^2.$$

Therefore Lemma 1 implies that

$$\|tx + (1-t)u - \lambda[ty + (1-t)v]\|^2 \geqslant 1+\lambda^2$$

which is contradictory with $\|x-\lambda y\|^2 < 1+\lambda^2$.

($P'_\lambda \Rightarrow Q_\varepsilon$). Let $x, y \in S$ be such that $\|x-y\| = \varepsilon$ and let

$$u = (4-\varepsilon^2)^{-1/2}(x+y), \qquad v = \varepsilon^{-1}(x-y).$$

Then $v \in S$ and

$$\|u+\lambda v\|^2 = \|u-\lambda v\|^2 = 4(4-\varepsilon^2)^{-1} = 1+\lambda^2.$$

Since the function $F(t) = \|tu+\lambda v\|$ is convex and such that $\lambda = F(0) < F(1) = F(-1) = (1+\lambda^2)^{1/2}$, it follows from Lemma 1 that $u \in S$, i.e., that $\|x+y\|^2 = 4-\varepsilon^2$.

($Q_\varepsilon \Rightarrow R_\varepsilon$). It is obvious.

($R_\varepsilon \Rightarrow Q_\varepsilon$). It is proved in [10] that in every real plane of E the set $\{x+y\colon x, y \in S, \|x-y\| = \varepsilon\}$ is a symmetric Jordan rectifiable curve which encloses $(4-\varepsilon^2)$ times the area enclosed by S. Then the existence of $x, y \in S$ such that $\|x-y\| = \varepsilon$ and $\|x+y\|^2 < 4-\varepsilon^2$ would imply the contradictory existence of $u, v \in S$ such that $\|u-v\| = \varepsilon$ and $\|u+v\|^2 > 4-\varepsilon^2$.

($Q_\varepsilon \Rightarrow T_\varepsilon$). It is an immediate corollary of Lemma 1.

($T_\varepsilon \Rightarrow P_\lambda$). Let $x, y \in S$ be such that $\|x+\lambda y\| = \|x-\lambda y\|$ and let

$$u = \|x+\lambda y\|^{-1}(x+\lambda y), \qquad v = \|x+\lambda y\|^{-1}(x-\lambda y).$$

Then $u, v \in S$ and

$$\|u-v\| = 2\lambda\, \|x+\lambda y\|^{-1}, \qquad \|u+v\| = 2\, \|x+\lambda y\|^{-1}.$$

If $\|x+\lambda y\|^2 < 1+\lambda^2$ then

$$\|u-v\| > 2\lambda(1+\lambda^2)^{-1/2} = \varepsilon, \qquad \|u+v\|^2 > 4(1+\lambda^2)^{-1} = 4-\varepsilon^2$$

which contradicts T_ε. And analogously for $\|x+\lambda y\|^2 > 1+\lambda^2$.

LEMMA 2. *If a real or complex normed linear space E satisfies the property P_λ ($\lambda > 0$), then in every real two-dimensional linear subspace of E there exist $x, y \in S$ such that*

$$\|x+\lambda y\| = \|x-\lambda y\| = \|\lambda x+y\| = \|\lambda x-y\| = (1+\lambda^2)^{1/2}.$$

Proof. We can suppose that E is the linear space $\mathbb{R}^2$ endowed with a norm and an orientation ω.

Let $s\colon \alpha \in [0, 2\pi] \to s(\alpha) \in S$ be the parametrization of S mentioned in Lemma 1. Elementary arguments prove that if E satisfies the property P_λ then for every $0 \leqslant \alpha \leqslant \pi$ there exists a unique $\alpha < g(\alpha) < \alpha+\pi$ such that

$$\|s(\alpha)+\lambda s(g(\alpha))\| = \|s(\alpha)-\lambda s(g(\alpha))\|, \qquad [s(\alpha), s(g(\alpha))] = \omega.$$

Also it is easy to see that the real function g is continuous and strictly increasing on $[0, \pi]$, and that if either $s(\alpha)+\lambda s(g(\alpha))=s(\beta)+\lambda s(g(\beta))$ or $\lambda s(\alpha)+s(g(\alpha))=\lambda s(\beta)+s(g(\beta))$ then $\alpha=\beta$.

Therefore the set

$$S_\lambda=\{x+\lambda y: x, y \in S, [x, y]=\omega, \|x+\lambda y\|=\|x-\lambda y\|\}$$

is the symmetric Jordan rectifiable curve $(1+\lambda^2)^{1/2} S$ and the set

$$S'_\lambda=\{\lambda x+y: x, y \in S, [x, y]=\omega, \|x+\lambda y\|=\|x-\lambda y\|\}$$

is also a symmetric Jordan rectifiable curve.

The area enclosed by S_λ is $(1+\lambda^2)$ times the area enclosed by S and, if we suppose that ω is the positive orientation of the plane, the area enclosed by S'_λ is given by

$$A(S'_\lambda)=\int_0^\pi [\lambda s_1(\alpha)+s_1(g(\alpha))]\, d[\lambda s_2(\alpha)+s_2(g(\alpha))]$$

$$-\int_0^\pi [\lambda s_2(\alpha)+s_2(g(\alpha))]\, d[\lambda s_1(\alpha)+s_1(g(\alpha))].$$

Taking into account the analogous formulae for $A(S_\lambda)$ and $A(S)$ we obtain that $A(S'_\lambda)=(1+\lambda^2)\, A(S)=A(S_\lambda)$. Thus $S_\lambda \cap S'_\lambda \neq \varnothing$ and there exist $x, y \in S$ such that

$$\|x+\lambda y\|^2=\|x-\lambda y\|^2=1+\lambda^2, \qquad \|\lambda x+y\|^2=1+\lambda^2, \qquad [x, y]=\omega.$$

Finally Proposition 1 assures that $\|\lambda x-y\|^2=1+\lambda^2$.

PROPOSITION 2. *If a real or complex normed linear space E satisfies the property P_λ for some $\lambda>0$ such that*

$$\lambda \notin D=\{\tan(k\pi/2n): n=2, 3, ...; k=1, 2, ..., n-1\}$$

then E is an inner product space.

Proof. Since E is an inner product space if and only if the same is true for all its real two-dimensional linear subspaces, we can consider E as the linear space $\mathbb{R}^2$ endowed with a norm.

Let $\lambda>0$. By Lemma 2 we can take $x, y \in S$ such that

$$\|x+\lambda y\|=\|x-\lambda y\|=\|\lambda x+y\|=\|\lambda x-y\|=(1+\lambda^2)^{1/2}.$$

Performing, if necessary, a linear transformation of the plane we can suppose that $x=s(0)=(1, 0)$, $y=s(\pi/2)=(0, 1)$, where s denotes the parametrization of S given in Lemma 1.

Then, if C is the unit euclidean circumference of the plane and if $0<\theta<\pi/2$ is such that $\lambda=\tan\theta$, the points

$$x_1=(1+\lambda^2)^{-1/2}(x+\lambda y)=s(\theta), \qquad y_1=(1+\lambda^2)^{-1/2}(-\lambda x+y)=s(\theta+\pi/2)$$

belong to $S\cap C$ and, taking into account Proposition 1, they are such that

$$\|x_1+\lambda y_1\|=\|x_1-\lambda y_1\|=\|\lambda x_1+y_1\|=\|\lambda x_1-y_1\|=(1+\lambda^2)^{1/2}.$$

The same arguments are valid to prove that the points

$$x_2=(1+\lambda^2)^{-1/2}(x_1+\lambda y_1)=s(2\theta),$$
$$y_2=(1+\lambda^2)^{-1/2}(-\lambda x_1+y_1)=s(2\theta+\pi/2)$$

also belong to $S\cap C$ and they are such that

$$\|x_2+\lambda y_2\|=\|x_2-\lambda y_2\|=\|\lambda x_2+y_2\|=\|\lambda x_2-y_2\|=(1+\lambda^2)^{1/2}.$$

Pursuing this process we obtain a set $\{x, x_1, x_2, ...\}$ which is contained in $S\cap C$ and, provided $\lambda\notin D$, is dense in S and C. Therefore $S=C$ and E is an inner product space.

Remark. If the unit sphere of a norm in $\mathbb{R}^2$ is invariant under rotations of angle $\pi/2n$ $(n=2, 3, ...)$ and if $\lambda=\tan(k\pi/2n)$ $(k=1, 2, ..., n-1)$, then such a normed linear space satisfies the property P_λ.

Thus for every $\lambda\in D$ there exist real two-dimensional noninner product spaces satisfying P_λ, for example, the linear space $\mathbb{R}^2$ endowed with the norm whose unit sphere is the regular $4n$-gon.

COROLLARY. *For* $\varepsilon\neq 2\cos(k\pi/2n)$ $(n=2, 3, ...; k=1, 2, ..., n-1)$, *the properties* Q_ε, R_ε, *and* T_ε *are characteristic of the inner product spaces.*

Nevertheless, for $\varepsilon=2\cos(k\pi/2n)$ there exist real two-dimensional non-inner product spaces satisfying the above properties.

Conjecture. The property P_λ, without restriction on $\lambda>0$, is characteristic of the inner product spaces of real dimension $\geqslant 3$.

PROPOSITION 3. *If a real or complex normed linear space E satisfies the property*

$$P_2\colon x, y\in S, \|x+2y\|=\|x-2y\| \Rightarrow \|x+2y\|^2=5$$

then E is an inner product space.

Proof. We only need to show that $2\notin D$.

Suppose there exist $n = 2, 3, \ldots$ and $k = 1, 2, \ldots, n-1$, such that $\tan(k\pi/2n) = 2$. Denoting $\theta = k\pi/2n$ and using a well-known trigonometrical formula we have that

$$\tan 2n\theta = \frac{\binom{2n}{1}\tan\theta - \binom{2n}{3}\tan^3\theta + \cdots \pm \binom{2n}{2n-1}\tan^{2n-1}\theta}{\binom{2n}{0}\tan^0\theta - \binom{2n}{2}\tan^2\theta + \cdots \mp \binom{2n}{2n}\tan^{2n}\theta} = 0.$$

Then it suffices to see that 2 is not a root of the polynomial

$$\binom{2n}{1} - \binom{2n}{3}x^2 + \binom{2n}{5}x^4 - \cdots \pm \binom{2n}{2n-1}x^{2n-2}$$

and this follows from the fact that if $2n = 2^p q$, with q odd, then for $k = 3, 5, \ldots, 2n-1$, $\binom{2n}{k} = 2^p r_k$, with r_k integer.

Remark. Following James [7] and Carlsson [5] we can say that x is λ-Isosceles orthogonal to y, $x \perp_{\lambda I} y$, when $\|x + \lambda y\| = \|x - \lambda y\|$ and that x is λ-Pythagorean orthogonal to y, $x \perp_{\lambda P} y$, when $\|x + \lambda y\|^2 = \|x\|^2 + \lambda^2 \|y\|^2$.

Then we can paraphrase Proposition 2 by saying that for every $\lambda \notin D$ the property (or its converse)

$$x, y \in S, \qquad x \underset{\lambda I}{\perp} y \Rightarrow x \underset{\lambda P}{\perp} y$$

is characteristic of the inner product spaces.

With this formulation Proposition 2 is in the line of many results of characterization of inner product spaces based in the relation between various types of generalized orthogonality in normed linear spaces [1, 6, 8, 9, 11].

ACKNOWLEDGMENT

Professor Dan Amir called our attention to the case $\dim E \geqslant 3$ and to the equivalence between property P_λ (the only one considered in the first version of this paper) and property R_ε. It is a pleasure to acknowledge those and other interesting hints.

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